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# New Formulation of Quantum Dynamical Entropies

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**Abstract**

Classical dynamical entropy is an important tool to analyse the efficiency of information transmission in communication processes.

Here we report new formulations of quantum dynamical entropy.

## Introduction

Classical dynamical (or Kolmogorov - Sinai) entropy $S(T)$ for a measure preserving transformation $T$ was defined on a message space through the measure of finite partitions of a measurable space. The classical coding theorems are the important tools to analyse communication processes, and they are formulated by the mean dynamical entropy and the mean dynamical mutual entropy. The mean entropy exhibits the amount of information per one letter for a signal sequence sending from the input source and the mean mutual entropy does the amount of information per one letter for a signal sequence trnsmitted from the input system to the output system.

Quantum dynamical entropy has been studied by Connes, Stormer [C.2], Emch [E.1], CNT (Connes, Narnhofer, Thirring) [C.1] and others [B.1,O.7].

Recently, quantum dynamical entropy and mutual entropy were introduced by the present author in terms of the complexity of information dynamics [O.8,M.1]. Furthermore, another formulation of the dynamical entropy through QMC was done in [A.4].

In §1, we review the formulation by CNT [C.1]. In §2, the formulation by the complexity is presented. In §3, the formulation through quantum Markov chain (QMC) [A.4] is discussed. In §4, we consider the relations among these formulations.

## §1. CNT Formulation

Let $(\mathcal{A},\theta_{\mathcal{A}},\varphi)$ be an initial C*-system. That is, $\mathcal{A}$ is a unital $C^*$-algebra, $\theta_{\mathcal{A}}$ is an automorphism of $\mathcal{A}$, and $\varphi$ is an invariant state over $\mathcal{A}$ with respect to $\theta_{\mathcal{A}}$; $\varphi\circ\theta_{\mathcal{A}}=\varphi$. Let $\mathcal{N}$ be a finite dimensional C* - subalgebra of $\mathcal{A}$. The CNT entropy [C.1] for a subalgebra $\mathfrak{N}$ is

$$H_{\varphi}(\mathcal{N})=\sup\left\{\sum_k \lambda_k S(\omega_k \mid \mathcal{N},\, \varphi\mid\mathcal{N})\,;\ \varphi=\sum_k \lambda_k\omega_k \ \text{ finite decomposition of } \varphi\right\}.$$

where $\varphi \mid \mathcal{N}$ is the restriction of a state $\varphi$ to $\mathcal{N}$ and $S(\cdot,\cdot)$ is the relative entropy for $C^*$-algebra [A.6,U.1,O.7].

The CNT dynamical entropy with respect to $\theta_{\mathcal{A}}$ and $\mathcal{N}$ is given by

$$\tilde{H}_{\varphi}(\theta_{\mathcal{A}}, \mathcal{N}) = \limsup_{N\to\infty} \frac{1}{N} H_{\varphi}(\mathcal{N} \vee \theta\mathcal{N} \vee \cdots \vee \theta^{N-1}\mathcal{N}).$$

The dynamical entropy for $\theta_{\mathcal{A}}$ is defined by

$$\tilde{H}_{\varphi}(\theta_{\mathcal{A}}) = \sup_{\mathcal{N}} \tilde{H}_{\varphi}(\theta_{\mathcal{A}}, \mathcal{N}),$$

This dynamical entropy is sometimes called the maximal average information gain with respect to $\theta_{\mathcal{A}}$.

## §2. Formulation by Complexity

In this section, we first review the concept of complexity, which are the key concepts of Information Dynamics (ID for short) introduced in [O.6,O.8,O.9].

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha(G))$ and $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}(\overline{\mathcal{A}}), \overline{\alpha}(\overline{G}))$ be an input (initial) and an output (final) C*-systems, respectively, where $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) is a unital $C^*$-algebra, $\mathfrak{S}(\mathcal{A})$ (resp. $\overline{\mathfrak{S}}(\overline{\mathcal{A}})$) is the set of all states on $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) and $\alpha(G)$ (resp. $\overline{\alpha}(\overline{G})$) is an automorphism of $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) indexed by a group $G$ (resp. $\overline{G}$).

A channel [O.1,O.4,O.6] is a map $\Lambda^*$ from $\mathfrak{S}(\mathcal{A})$ to $\overline{\mathfrak{S}}(\bar{\mathcal{A}})$.

For a w*- compact convex subset $\mathcal{S}$ of $\mathfrak{S}$, there exists a measure $\mu$ with the barycenter $\varphi$ such that

$$\varphi = \int_S \omega \ d\mu$$

The compound state introduced in [O.2,O.3] exhibiting the correlation between an initial state $\varphi$ and its final state $\Lambda^*\varphi$ is given by

$$\mathcal{E}^*\varphi = \int_S \omega \otimes \Lambda^*\omega \ d\mu$$

This compound state corresponds with the joint measure in classical systems.

There are two complexities in ID. One is a complexity $C_T^S(\varphi)$ of a system itself and another is a transmitted complexity $T^S(\varphi\,;\Lambda^*)$ from an initial system to a final system. These complexities should satisfy the following conditions:

(i) $\forall\varphi \in \mathcal{S} \subset \mathfrak{S}$

$$C^S(\varphi) \geq 0, \quad T^S(\varphi\,;\Lambda^*) \geq 0,$$

(ii) If there exists a bijection $j : ex\mathfrak{S} \to ex\mathfrak{S}$, the set of all extreme points in $\mathfrak{S}$, then

$$C^{j(S)}(j(\varphi)) = C^S(\varphi)$$
$$T^{j(S)}(j(\varphi)\,;\Lambda^*) = T^S(\varphi\,;\Lambda^*)$$

(iii) Let $\Psi = \varphi \otimes \psi \in \mathcal{S}_t$ and $\varphi \in \mathcal{S}$, $\psi \in \overline{\mathcal{S}}$. Then

$$C^{\mathcal{S}_t}(\Phi) = C^{\mathcal{S}}(\varphi) + C^{\overline{\mathcal{S}}}(\psi)$$

(iv) $0 \leq T^{\mathcal{S}}(\varphi\,;\Lambda^*) \leq C^{\mathcal{S}}(\varphi)$
(v) $T^{\mathcal{S}}(\varphi\,; id) = C^{\mathcal{S}}(\varphi)$

Instead of (iii) above, when for $\Phi \in \mathcal{S}_t \subset \mathfrak{S}_t = \mathfrak{S} \otimes \overline{\mathfrak{S}}$ and $\varphi \equiv \Phi|\mathcal{A}$, $\psi \equiv \Phi|\overline{\mathcal{A}}$

$$C^{\mathcal{S}_t}(\Phi) \leq C^{\mathcal{S}}(\varphi) + C^{\overline{\mathcal{S}}}(\psi)$$

is satisfied, we call $\{C^{\mathcal{S}}, T^{\mathcal{S}}\}$ is a pair of strong complexities. These complexities generalize several expressions of chaos [O.10].

Let us explain the formulation of three types of entropic complexity introduced in [O.2].

Let $(\mathcal{A},\ \mathfrak{S}(\mathcal{A}),\ \alpha(G))$, $(\overline{\mathcal{A}},\ \overline{\mathfrak{S}}(\overline{\mathcal{A}}),\ \overline{\alpha}(\overline{G}))$ and $\mathcal{S}$ as before. Let $M_\varphi(\mathcal{S})$ be the set of all maximal measures $\mu$ on $\mathcal{S}$ with the fixed barycenter $\varphi$ and let $F_\varphi(\mathcal{S})$ be the set of all measures of finite support with the fixed barycenter $\varphi$. Then we have three pairs of complexities such as

$$\begin{aligned}
&T^{\mathcal{S}}(\varphi\,;\Lambda^*) \equiv \sup\left\{\int_{\mathcal{S}} S(\Lambda^*\omega,\Lambda^*\varphi)\,d\mu\,;\ \mu \in M_\varphi(\mathcal{S})\right\} \\
&C_T^{\mathcal{S}}(\varphi) \equiv T^{\mathcal{S}}(\varphi\,;\ id) \\
&I^{\mathcal{S}}(\varphi\,;\Lambda^*) \equiv \sup\left\{S\left(\int_{\mathcal{S}} \omega\otimes\Lambda^*\omega d\mu,\ \varphi\otimes\Lambda^*\varphi\right);\ \mu \in M_\varphi(\mathcal{S})\right\} \\
&C_I^{\mathcal{S}}(\varphi) = I^{\mathcal{S}}(\varphi\,;\ id) \\
&J^{\mathcal{S}}(\varphi\,;\Lambda^*) \equiv \sup\left\{\int_{\mathcal{S}} S(\Lambda^*\omega,\Lambda^*\varphi)\,d\mu_f;\ \mu_f \in F_\varphi(\mathcal{S})\right\} \\
&C_J^{\mathcal{S}}(\varphi) \equiv J^{\mathcal{S}}(\varphi\,;\ id)\,.
\end{aligned}$$

Based on the above complexities, we can formulate the quantum dynamical entropy [O.2,O.8,O.9]: Let $\theta_{\mathcal{A}}$ (resp. $\theta_{\mathcal{B}}$) be a stationary (invariant) automorphism of $\mathcal{A}$ (resp. $\mathcal{B}$); $\varphi\circ\theta_{\mathcal{A}} = \varphi, \psi\circ\theta_{\mathcal{B}} = \psi$, and $\Lambda^*$ be a covariant channel (i.e., $\Lambda\circ\theta_{\mathcal{A}} = \theta_{\mathcal{B}}\circ\Lambda$) from $\mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}(\mathcal{A})$. $\mathcal{A}_k$ (resp. $\mathcal{B}_k$) is a finite subalgebra of $\mathcal{A}$ (resp. $\mathcal{B}$). Moreover, let $\alpha_k$ (resp. $\beta_k$) be a completely positive unital map from $\mathcal{A}_k$ (resp. $\mathcal{B}_k$) to $\mathcal{A}$ (resp. $\mathcal{B}$) and $\alpha^M$ and $\beta_\Lambda^N$ be

$$\begin{aligned}
\alpha^M &= (\alpha_1,\ \alpha_2,\ \cdots,\alpha_M)\,, \\
\beta_\Lambda^N &= (\Lambda\circ\beta_1,\ \Lambda\circ\beta_2,\ \cdots,\Lambda\circ\beta_N)
\end{aligned}$$

Two compound states for $\alpha^M$ and $\beta_\Lambda^N$ with respect to $\mu \in M_\varphi(\mathcal{S})$ are defined as

$$\begin{aligned}
\Phi_\mu^{\mathcal{S}}(\alpha^M) &= \int_{\mathcal{S}} \bigotimes_{m=1}^{M} \alpha_m^*\omega d\mu, \\
\Phi_\mu^{\mathcal{S}}(\alpha^M \cup \beta_\Lambda^N) &= \int_{\mathcal{S}} \bigotimes_{m=1}^{M} \alpha_m^*\omega \bigotimes_{n=1}^{N} \beta_n^*\Lambda^*\omega d\mu.
\end{aligned}$$

By using these compound states, we define three transmitted complexities [O.8]:

$$T_{\varphi}^{S}\left(\alpha^{M},\, \beta_{\Lambda}^{N}\right)$$
$$\equiv \sup\{\int_{S} S(\underset{m=1}{\overset{M}{\otimes}} \alpha_{m}^{*}\omega \underset{n=1}{\overset{N}{\otimes}} \beta_{n}^{*}\Lambda^{*}\omega,\ \Phi_{\mu}^{S}(\alpha^{M}) \otimes \Phi_{\mu}^{S}(\beta_{\Lambda}^{N}))d\mu;\ \mu \in M_{\varphi}(S)\}$$
$$I_{\varphi}^{S}(\alpha^{M},\, \beta_{\Lambda}^{N}) \equiv \sup\{S(\Phi_{\mu}^{S}(\alpha^{M} \cup \beta_{\Lambda}^{N}),\ \Phi_{\mu}^{S}(\alpha^{M}) \otimes \Phi_{\mu}^{S}(\beta_{\Lambda}^{N}))\ ;\ \mu \in M_{\varphi}(\mathcal{S})\}$$
$$J_{\varphi}^{S}(\alpha^{M},\, \beta_{\Lambda}^{N})$$
$$\equiv \sup\{\int_{S} S(\underset{m=1}{\overset{M}{\otimes}} \alpha_{m}^{*}\omega \underset{n=1}{\overset{N}{\otimes}} \beta_{n}^{*}\Lambda^{*}\omega,\ \Phi_{\mu}^{S}(\alpha^{M}) \otimes \Phi_{\mu}^{S}(\beta_{\Lambda}^{N}))d\mu_{f};\ \mu_{f} \in F_{\varphi}(\mathcal{S})\}$$

When $\mathcal{A}_k = \mathcal{A}_0 = \mathcal{B}_k$, $\mathcal{A} = \mathcal{B}$, $\theta_{\mathcal{A}} = \theta_{\mathcal{B}} = \theta$, $\alpha_k = \theta^{k-1} \circ \alpha = \beta_k$, the mean transmitted complexity is

$$\tilde{T}_{\varphi}^{S}\left(\theta, \alpha,\, \Lambda^{*}\right) \equiv \limsup_{N\to\infty} \frac{1}{N} T_{\varphi}^{S}\left(\alpha^{N},\, \beta_{\Lambda}^{N}\right)$$
$$\tilde{T}_{\varphi}^{S}\left(\theta,\, \Lambda^{*}\right) \equiv \sup_{\alpha} \tilde{T}_{\varphi}^{S}\left(\theta, \alpha,\, \Lambda^{*}\right)$$

Same for $\tilde{I}_{\varphi}^{S}, \tilde{J}_{\varphi}^{S}$. These quantities have the similar properties as the CNT entropy [O.8, M.1].

## §3. Formulation by QMC

Another formulation of the dynamical entropy is due to quantum Markov chain [A.4].

Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\varphi$ be a stationary faithful normal state on $\mathcal{A}$ and $\mathcal{A}_0 = M_d$ ($d \times d$ matrix algebra). Take the transition expectation $\mathcal{E}_{\gamma} : \mathcal{A}_0 \otimes \mathcal{A} \to \mathcal{A}$ of Accardi [A.1,A.2] such that

$$\mathcal{E}_{\gamma}(\tilde{A}) = \sum_{i} \gamma_i A_{ii} \gamma_i$$

where $\tilde{A} = \sum_{i,j} e_{ij} \otimes A_{ij} \in \mathcal{A}_0 \otimes \mathcal{A}$ and $\gamma = \{\gamma_j\}$ is a finite partition of unity $I \in \mathcal{A}$. For a state $\varphi$ on $\mathcal{A}$, the quantum Markov chain $\psi \equiv \{\varphi, \mathcal{E}_{\gamma,\theta}\} \in \mathfrak{S}(\overset{\infty}{\underset{1}{\otimes}} \mathcal{A}_0)$ is defined by

$$\psi(j_1(A_1) \cdots j_n(A_n))$$
$$\equiv \varphi(\mathcal{E}_{\gamma,\theta}(A_1 \otimes \mathcal{E}_{\gamma,\theta}(A_2 \otimes \cdots \otimes A_{n-1}\mathcal{E}_{\gamma,\theta}(A_n \otimes I)\cdots)))$$

for each $n \in \mathbb{N}$ and each $A_1, \cdots, A_n \in \mathcal{A}_0$, where $\mathcal{E}_{\gamma,\theta} = \theta \circ \mathcal{E}_{\gamma}$, $\theta \in Aut(\mathcal{A})$, $\psi| \overset{n}{\underset{1}{\otimes}} \mathcal{A}_0 \equiv \psi_n$ and $j_k$ is the embeding from $\mathcal{A}_0$ to $\overset{\infty}{\underset{1}{\otimes}} \mathcal{A}_0$ such as $j_k(A) = I \otimes \cdots \otimes I \otimes \underset{k\cdot th}{A} \otimes I \cdots$. For our $\mathcal{E}_{\gamma,\theta}$, $\psi$ is written as

$$\psi(j_1(A_1) \cdots j_n(A_n)) = \psi_{[0,n]}(A_1 \otimes \cdots \otimes A_n \otimes I)$$
$$= \psi_n(A_1 \otimes \cdots \otimes A_n),$$

where $\psi_{[0,n]}$ and $\psi_n$ are the faithful normal states on $\overset{n}{\underset{1}{\otimes}}\mathcal{A}_0 \otimes \mathcal{A}$ and $\overset{n}{\underset{1}{\otimes}}\mathcal{A}_0$, respectively. When $\varphi$ is defined by a trace class operator $\rho$ such that $\varphi(\ \cdot\ ) = tr\rho\ \cdot$ , the density operators $\rho_{[0,n]}$ and $\xi_n$ of $\psi_{[0,n]}$ and $\psi_n$ are given by

$$\rho_{[0,n]} = \sum_{i_1}\cdots\sum_{i_n} e_{i_1i_1}\otimes\cdots\otimes e_{i_ni_n}\otimes\theta^n(\gamma_{i_n})\cdots\gamma_{i_1}\rho\gamma_{i_1}\cdots\theta^n(\gamma_{i_n})$$

$$\xi_n \equiv \sum_{i_1}\cdots\sum_{i_n} tr_{\mathcal{A}}(\theta^n(\gamma_{i_n})\cdots\gamma_{i_1}\rho\gamma_{i_1}\cdots\theta^n(\gamma_{i_n}))e_{i_1i_1}\otimes\cdots\otimes e_{i_ni_n}$$

Take

$$P_{i_n\cdots i_1} = tr_{\mathcal{A}}(\theta^n(\gamma_{i_n})\cdots\gamma_{i_1}\rho\gamma_{i_1}\cdots\theta^n(\gamma_{i_n})$$

The mean dynamical entropy [A.4] through QMC is

$$\begin{aligned}\tilde{S}_\varphi(\theta;\gamma) &\equiv \lim_{n\to\infty}\frac{1}{n}S_n(\gamma,\theta)\\ &= -\lim_{n\to\infty}\frac{1}{n}tr\xi_n\log\xi_n\\ &= -\lim_{n\to\infty}\frac{1}{n}\sum_{i_1,\cdots,i_n}P_{i_n\cdots i_1}\log P_{i_n\cdots i_1},\end{aligned}$$

When $P_{i_n\cdots i_1}$ satisfies the Markov property, the above equation becomes

$$\tilde{S}_\varphi(\theta;\gamma) = -\sum_{i_1,i_2}P(i_2|i_1)P(i_1)\log P(i_2|i_1).$$

The dynamical entropy through QMC with respect to $\theta$ and a subalgebra $\mathcal{A}_1$ of $\mathcal{A}$ is

$$\tilde{S}_\varphi(\theta;\mathcal{A}_1) \equiv \sup\{\tilde{S}_\varphi(\theta;\gamma);\gamma\subset\mathcal{A}_1\}.$$

## §4. Relations Among Three Formulations

In tis section, we discuss the relations among the above three formulations. The $\mathcal{S}$ - mixing entropy in GQS (general quantum systems) introduced in [O.5] is

$$S^{\mathcal{S}}(\varphi) = \inf\{H(\mu)\ ;\ \mu\in M_\varphi(\mathcal{S})\},$$

where $H(\mu)$ is given by

$$H(\mu) = \sup\left\{-\sum_{A_k\in\tilde{A}}\mu(A_k)\log\mu(A_k)\ :\ \tilde{A}\in P(\mathcal{S})\right\}$$

where $P(\mathcal{S})$ is the set of all partitions of $\mathcal{S}$.

The following theorem [O.8,M.1] shows the relation between the CNT formulation and the formulation by complexity.

**Theorem 4.1** Under the above settings, we have the following relations:
(1) $0 \le I^{S}(\varphi\,;\Lambda^*) \le T^{S}(\varphi\,;\Lambda^*) \le J^{S}(\varphi\,;\Lambda^*)$
(2) $C_I^{\mathfrak{S}}(\varphi) = C_T^{\mathfrak{S}}(\varphi) = C_J^{\mathfrak{S}}(\varphi) = S^{\mathfrak{S}}(\varphi) = H_\varphi(\mathcal{A})$
(3) $\mathcal{A} = \bar{\mathcal{A}} = B(\mathcal{H})$, for any density operator $\rho$

$$0 \le I^{S}(\rho\,;\Lambda^*) = T^{S}(\rho;\Lambda^*) \le J^{S}(\rho\,;\Lambda^*)$$

Since there exists a model showing $S^{I(\alpha)}(\varphi) > H_\varphi(\mathcal{A}_\alpha)$, $S^{S}(\varphi)$ distinguishes states more sharply than $H_\varphi(\mathfrak{N})$, where $\mathcal{A}_\alpha = \{A \in \mathcal{A};\ \alpha(A) = A\}$.

Moreover we have [O.9].

(1) When $\mathcal{A}_n, \mathcal{A}$ are the abelian C*-algebras and $\alpha_k$ is an embedding map, then

$$T^{\mathfrak{S}}(\mu;\alpha^M) = S_\mu^{\text{classical}}\left(\bigvee_{m=1}^{M} \tilde{A}_m\right)$$

$$I^{\mathfrak{S}}(\mu;\alpha^M,\beta^N) = I_\mu^{\text{classical}}\left(\bigvee_{n=1}^{M} \tilde{A}_n, \bigvee_{n=1}^{N} \tilde{B}_n\right)$$

are satisfied for any finite partitions $\tilde{A}_n, \tilde{B}_n$ on the probability space $(\Omega = \text{spec}(\mathcal{A}), \mathcal{F}, \mu)$.

(2) When $\Lambda$ is the restriction of $\mathcal{A}$ to a subalgebra $\mathcal{M}$ of $\mathcal{A}$ ; $\Lambda = |\mathcal{M}$ and

$$\mathcal{N} \subset \mathcal{A}_0, \mathcal{A} = \overset{\mathrm{N}}{\otimes} \mathcal{A}_0, \theta \in Aut(\mathcal{A});$$

$$\alpha^N \equiv (\alpha, \theta\circ\alpha, \cdots; \theta^{N-1}\circ\alpha);$$

$$\alpha = \beta; \mathcal{A}_0 \to \mathcal{A} \text{ embedding};$$

$$\mathcal{N}_N \equiv \overset{N}{\underset{1}{\otimes}} \mathcal{N},$$

we have

$$H_\varphi(\mathcal{M}) = J^{\mathfrak{S}}(\varphi;|\mathcal{M}) = J_\varphi^{\mathfrak{S}}(id;|\mathcal{M}),$$

$$\tilde{H}_\varphi(\theta;\mathcal{N}) = \tilde{J}_\varphi^{\mathfrak{S}}(\theta;\mathcal{N}) = \limsup_{N\to\infty} \frac{1}{N} J_\varphi^{\mathfrak{S}}(\alpha^N;|\mathcal{N}_N).$$

We show the relation between the formulation by complexity and the formulation by QMC.

When $\varphi$ is defined by a trace class operator $\rho$ such that $\varphi(\,\cdot\,) = tr\rho\,\cdot$ , we define a map $\mathcal{E}_{(n)}^*$ from $\mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}((\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0)\otimes\mathcal{A})$ by

$$\mathcal{E}_{(n)}^*(\varphi)(A) = tr\sum_{i_1}\cdots\sum_{i_n} e_{i_1 i_1}\otimes\cdots\otimes e_{i_n i_n}\otimes\theta^n(\gamma_{i_n})\cdots\gamma_{i_1}\rho\gamma_{i_1}\cdots\theta^n(\gamma_{i_n})A$$

for any $A \in (\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0) \otimes \mathcal{A}$. Take a map $E_{(n)}$ from $\mathfrak{S}((\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0) \otimes \mathcal{A})$ to $\mathfrak{S}(\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0)$ such that

$$(E_{(n)}\omega)(Q) = \omega(Q \otimes I), \quad \forall Q \in \overset{n}{\underset{1}{\otimes}} \mathcal{A}_0$$

Then a channel $\Gamma^*_{(n)}$ from $\mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}(\overset{n}{\underset{1}{\otimes}} \mathcal{A}_0)$ is given by

$$\Gamma^*_{(n)} \equiv E_{(n)} \circ \mathcal{E}^*_{(n)}$$

so that $\Gamma^*_{(n)}(\varphi)(A) = tr\xi_n A$ for any $A \in \overset{n}{\underset{1}{\otimes}} \mathcal{A}_0$ and

$$\tilde{S}_\varphi(\theta;\gamma) = \lim_{n\to\infty} \frac{1}{n} C_I^{\mathfrak{S}}(\Gamma^*_{(n)}\varphi) = \tilde{C}_I(\varphi;\theta,\gamma).$$

In any case, the formulation by the entropic complexities contains other formulations, moreover it opens other possibility to classify the dynamical systems more fine [A.5].

# References


[A.1] L. Accardi, Noncommutative Markov chains, In internatinal School of Mathematical Physics, Camerino, pp. 268–295, 1974.

[A.2] L. Accardi, A. Frigerio and J. Lewis, Quantum stochastic processes, Publications of the Research institute for Mathematical Sciences Kyoto University, **18**, pp.97–133, 1982.

[A.3] L. Accardi and M. Ohya, Compound channels, transition expectations and liftings", to appear in J. Multivariate Analysis.

[A.4] L. Accardi, M. Ohya and N. Watanabe, Kolmogorov Sinai entropy through quantum Markov chain, to appear in Open System and Information Dynamics.

[A.5] L. Accardi, M. Ohya and N. Watanabe, Note on quantum dynamical entropy, to appear in Reports on Mathematical Physics.

[A.6] R. Alicki and M. Fannes, Defining quantum dynamical entropy, Letters in Math. Physics, **32**, 75–82, 1994.

[A.7] H. Araki, Relative entropy for states of von Neumann algebras, Publ. RIMS Kyoto Univ. **11**, pp. 809–833, 1976.

[B.1] F. Benatti, Deterministic chaos in infinite quantum systems, Springer - Verlag, 1993.

[B.2] L. Bilingsley, Ergodic Theory and Information, Wiley, New York, 1965.

[C.1] A. Connes, H. Narnhoffer and W. Thirring, Dynamical entropy of $C^*$algebras and von Neumann algebras, Commun. Math. Phys., **112**, pp.691-719, 1987.

[C.2] A. Connes and E. Størmer, Entropy for automorphisms of $\mathrm{II}_1$ von Neumann algebras, Acta Math. **134**, pp. 289–306, 1975.

[E.1] G.G. Emch, Positivity of the K - entropy on non - abelian K - flows, Z. Wahrscheinlichkeitstheory verw. Gebiete **29**, pp.241–252, 1974.

[F.1] L. Feinstein, Foundations of Information Theory, Macgrow-Hill, 1965.

[K.1] A.N. Kolmogorov, Theory of transmission of information, Amer. Math. Soc. Translation, Ser.2, **33**, pp. 291, 1963.

[M.1] N. Muraki and M. Ohya, Entropy functionals of Kolmogorov Sinai type and their limit theorems", to appear in Letter in Mathematical Physics.

[N.1] J. von Neumann, Die Mathematischen Grundlagen der Quanten - mechanik, Springer - Berlin, 1932.

[O.1] M. Ohya, Quantum ergodic channels in operator algebras, J. Math. Anal. Appl. **84**, pp. 318–327, 1981.

[O.2] M. Ohya, On compound state and mutual information in quantum information theory, IEEE Trans. Information Theory, **29**, pp. 770-777, 1983.

[O.3] M. Ohya, Note on quantum probability, L. Nuovo Cimento, **38**, pp. 402–406, 1983.

[O.4] M. Ohya, Entropy transmission in C*-dynamical systems, J. Math. Anal. Appl., **100**, pp. 222–235, 1984.

[O.5] M. Ohya, Some aspects of quantum information theory and their applications to irreversible processes, Rep. Math. Phys., **27**, pp. 19–47, 1989.

[O.6] M. Ohya, Information dynamics and its application to optical communication processes, Lecture Notes in Physics **378**,Springer, pp. 81–92, 1991.

[O.7] M. Ohya and D. Petz, Quantum Entropy and its Use, Springer-Verlag, 1993.

[O.8] M. Ohya, State change, complexity and fractal in quantum systems", Quantum Communications and Measurement, pp. 309–320, 1995.

[O.9] M. Ohya, Foundation of entropy, complexity and fractal in quantum systems, to appear

[O.10] M. Ohya and N. Watanabe, Note on Irreversible Dynamics and Quantum Information, to appear in the Alberto Frigerio conference proceedings.

[U.1] A. Uhlmann, Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory, Commun. Math. Phys. **54**, pp. 21–32, 1977.